Review Article

Volume 4, Issue 1, pp 142-151, June 2022

ACHIEVERS JOURNAL OF SCIENTIFIC RESEARCH

Opecn Access Publications of Achievers University, Owo

Available Online at www.achieversjournalofscience.org

A Simple Way to View Mathematical Modelling

Adetula, L.O.[1]

[1]Department of Mathematical Sciences, Achievers University Owo, Nigeria.

Email: lawaladetula@achievers.edu.ng

Submitted: April 12, 2022 Revised: May 30, 2022 Accepted: June 2, 2022 Published: June 30, 2022

Abstract

This paper defines Mathematical Modelling as the art of translating or converting a real world system or situation (problem) into a mathematical form or statement. That is, Mathematical Modelling is the process of how to use mathematics in solving problems of real life situations. Whereas, Mathematical Model is a simplified mathematical representation (equations, graphs, functions, relations) of the basic characteristics of the real situation. This is the product of the process. That is, the real life problem that we come across is transformed into mathematical statements and then solved using mathematical techniques. Few examples (in particular, the model of Planetary system) are used to concretize this concept of Mathematical Modelling. It is also demonstrated that to some extent, Mathematical Modelling is not only related to problem solving but a potent problem solving tool.

Keywords: Mathematical Modelling; Planetary System; Bode's Number

1.0 Introduction

A Mathematical model describes a system by a set of variables and a set of equations that establish relationships between the variables. However, the process of developing a Mathematical model is termed Mathematical Modeling. That is, a Mathematical modeling is a technique for describing and understanding the dynamics of a system using mathematical concepts and language. (Glenn, 2008).

Trivial examples of model (in this case physical model) that are in simple form are within our reach. For example, children meeting the model of reality (cars, aeroplanes, trains, balls, ships, babies, etc.) in their toys. Another simple example is when an architect exemplifies the features of a reality building in a building plan. Clearly, these are examples (making representation of real situation on a smaller scale) on how the models appear in our daily lives.

2.0 Model of Planetary Motion

The model of a Planetary system is used as concrete example to illuminate the concept of Mathematical Modeling. For easy understanding, this concept is segmented into at least four (4) steps.

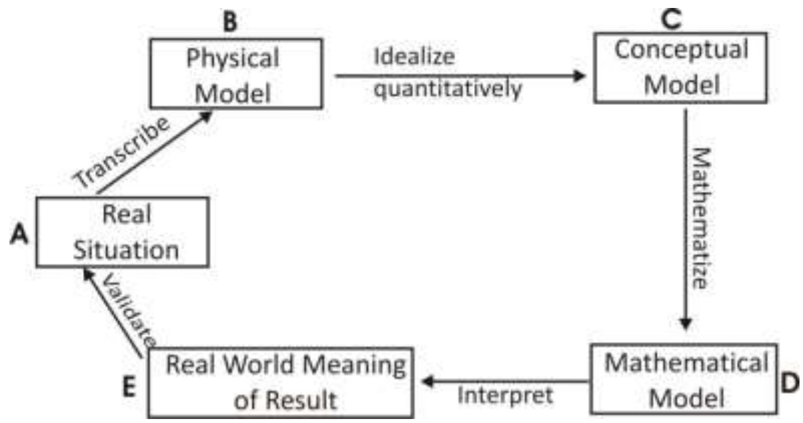

Fig 1: Mathematical Modeling Cycle

A. The group of all Planets revolving round the Sun as captured and viewed on the giant telescope will represent the *real situation*. Note that the real situation can only be fully captured depending on the effectiveness and efficiency of the giant telescope or aerial camera.

B. Transcribing or mapping this real situation as viewed into drawing on a paper, then this map (probably to scale) becomes the *physical model* (just like toys or building plan described above) of the real situation (Planetary system).

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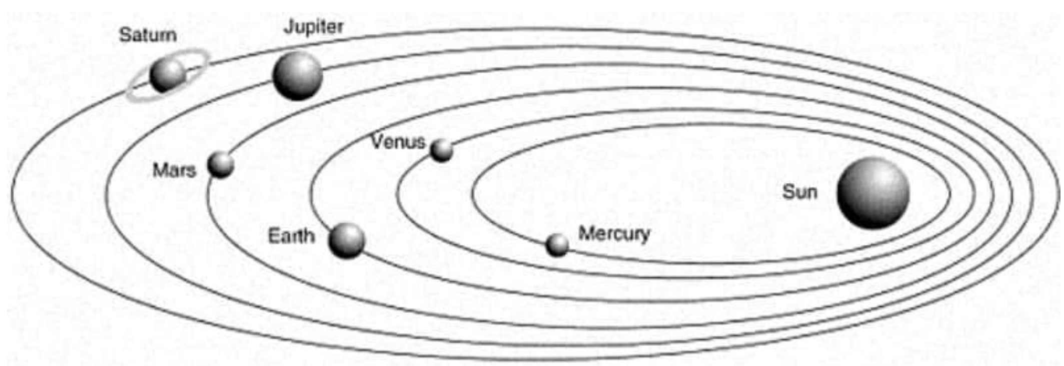


Fig 2: Planetary System

The historical perspective of the model of planetary system is very interesting; therefore, one or two things must be said on this. (Adetula, 2006). The Mathematicians before 500BC have discovered that this world was strictly governed and determined by mathematical laws. Then, the mathematical models to generate these laws were sought after.

From common sense, backed by religious beliefs, the Earth was taken to be at the centre of the revolving Planets to give the underlying or unifying theory of their motions. This was the geocentric theory (model) of Hipparchus (125BC) and Ptolemy (168BC). Unfortunately, the predictions of physical events, e.g. eclipse of the Sun, using this theory (geocentric mathematical model) were never accurate.

Clearly, there was lack of agreement between theoretical mathematical model and experimental measurement. Hence better theory (model) was sought after to be developed.

By 16th century, another theoretical system (model) of Planetary motion (a paradigm shift that offered radical simplification) by Copernicus and Kepler was conceived.

In this system, the Sun is at the centre (relatively fixed) and each Planet (in our solar system) moves in an ellipse about the Sun. This theoretical system is called heliocentric.

Fortunately, the predictions of natural (physical) events (e.g. eclipse of the Sun or Moon, appearance of the comets etc.) based on heliocentric mathematical model (theory) were profoundly accurate, superb triumph and success galore (Adetula, 2006). For example, using this theory, it is discovered that an eclipse of the Sun (solar eclipse) occurs when Sun, Moon and Earth become aligned. This situation of this perfect alignment is called syzygy. (Adetula, 2006).

With this heliocentric Mathematical model (theory), man is able to know that the movements of these heavenly bodies are regular and periodic, and is governed by mathematical laws.

C. How these Planets in the system are quantitatively related (e.g. as in each Planets' distance from the Sun, here distance can be considered as a variable), and other quantitative ideas to explain the working of the system. These quantitatively related ideas will be regarded as *Conceptual Model.*

According to Adetula (2004), a good example of this conceptual model is the one given by Elert Bode (1772). He studied the relative distances from the Sun to the Planets – (Mecury, Venus, Earth, Mars, Jupiter, and Saturn) known at that time.

First, after studying or gleening on the massive data collected on these relative distances, he conjectured a simple sequence of numbers (called Bodes numbers) thus:

0, 3, 6, 12, 24, 48, 96,

4 is added to each of these Bode's number to obtain the sequence

4, 7, 10, 16, 28, 52, 100.

This last sequence (except 5^{th} term "28") form the relative distances from the Sun to the known Planets. For instance, the ratio of the distance of Mars from the Sun compared to the distance of Venus from the Sun is 16:7

Though no Planet corresponded to the number 28 but with serious searching, the Asteroids Ceres was found in 1801 by Guiseppe Piazzi in the position of this searched Planet. It was concluded that they were the remains of the exploded Planet.

To seek for more Planets, extension of Bode's numbers to 192, 384 were sought and then the sequence extended to 196, 388. After a long search, it was interested to discover that 196 and 388 were correct for Uranus and for Pluto respectively.

Unfortunately, another Planet Neptune which is between Uranus and Pluto and corresponding to number 300 was discovered. This was not in the sequence. For this reason, Bode's law cannot be accepted as a law of nature.

Table 2: Relative and Actual Distance of Planets from the Sun

Planet	Mercury	Venus	Earth	Mars		Jupiter	Saturn	Uranus	Neptune	Pluto
Relative Distance from the Sun	4	7	10	16	28	52	100	196	300	388
Actual Distance from the Sun (in km)	?	?	150×10^6	?		?	?	?		?

Nevertheless, this Bode's sequence is very helpful to calculate the actual distance of any other Planets to the Sun, once you know the actual distance from Earth to the Sun (which we know to be approximately 150×10^6km.

For example: Find the distance from Uranus to the Sun.

Let x (in km) represent the distance from Uranus to the Sun.

Since the relative distance from Uranus to the Sun compared to the distance of Earth from the Sun is 196 to 10

Therefore, $\frac{x}{150 \times 10^6} = \frac{196}{10}$ (using proportion)

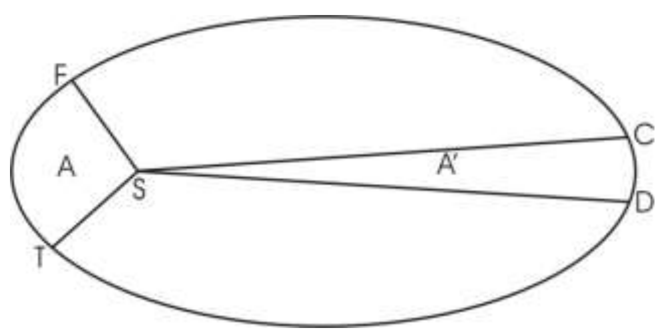
$\therefore \quad x = 196 \times 15 \times 10^6 = 2.94 \times 10^9$km

D. Generating set of equations (or laws) that take into account many quantitative factors in the system that represent the real situations. This is called *Mathematical Model.*

Three of such equations (laws) are considered by Kepler a German Mathematician and astronomer (Adetula, 2012).

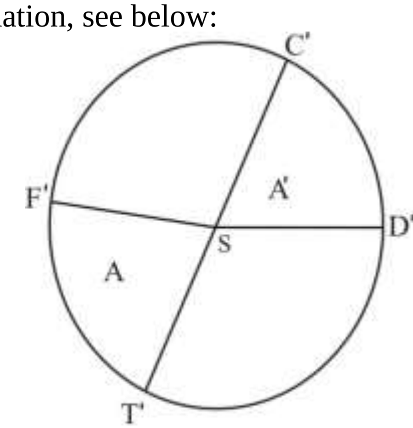
First Law: Every Planet's orbit is an ellipse, with the Sun at the focus. This indicated that Kepler helped Copernicus to refine the heliocentric model with the introduction of elliptical orbits instead of circular orbit that cannot explain why Planets orbited the Sun at different speed at different time.

Second Law: A line joining the Sun and a Planet sweeps out equal areas in equal time:



That is, if S is the position of the Sun at the focus of the Planet's elliptical orbit, then for a given time t, if the area A swept by the Planet from point T to F from the Sun S, is equal to the area A′ swept by the same Planet from C to D (that is A = A′, and arc TF is longer than arc CD); then the motion of the Planet will be faster between point T and F, and will be slower between point C and D, because the same time t is used to cover each of the arcs.

Unlike circular orbit, this shows that the speed of the Planet cannot be uniform in the elliptical orbit. Clearly this agreed with observation. For more clarification and explanation, see below:



Copernicles (as stated in the First law) allowed the Sun (S) to be at the centre of the planetary system. However, he used circular orbits for the path of the planets. If T′, F′ and C′, D′ are positions of a Planet and if Area A (Sector S T′ F′) equals Area A′ (Sector S C′ D′), then from circle theorem, chord C′ D′ = chord T′ F′. Therefore, the same speed by the planet is used to cover chord C′ D′ and chord T′, F′. Hence the speed is uniform. Unfortunately, from observation, speed of Planets is not uniform in their orbital paths, hence Kepler came to rescue the situation satisfactorily by proposing elliptical instead of circular sorbits for the Planets as indicated above.

Third Law: There is a precise Mathematical relationship between Planet's distance (a) from the Sun and amount of time (p) it takes to revolve round the Sun. the relationship is $p^2 = ka^3$; *that is, period squared is directly proportional to distance cubed*. The constant of proportionality k, is 1. Hence, the relationship is $p^2 = a^3$

For example, how long will it take Uranus to revolve round the Sun if it takes 1 year (365 days) for the Earth?
Let x (in years) represent how long it will take Uranus.

Table 3: Relative Distance and Period (P) for Planets to Revolve Round the Sun

Planet	Mercury	Venus	Earth	Mars		Jupiter	Saturn	Uranus	Neptune	Pluto
Relative Distance from the Sun	4	7	10	16	28	52	100	196	300	388
Period (P) - Time to revolve round the Sun (in years)	?	?	1	?		?	?	?	?	?

∴ ratio of Uranus to Earth to revolve round the Sun is $\frac{x}{1}$.
But ratio of Uranus to Earth using Bode's number of Planet's distance from the Sun is $\frac{196}{10}$.

Hence, the precise Mathematical relationship of these two ratios is

$$\left(\frac{x}{1}\right)^2 = \left(\frac{196}{10}\right)^3 \Rightarrow \quad x^2 = (19.6)^3 = 7530 \Rightarrow x = 86\ years$$

For Jupiter, how long will it take this Planet to revolve round the Sun?
Again, let x (in years) represent how long it will take Jupiter.

Applying the appropriate proportion

Then; $\left(\frac{x}{1}\right)^2 = \left(\frac{52}{10}\right)^3 \Rightarrow \quad x^2 = (5.2)^3 = 141 \Rightarrow x = 12\ years$

For Mercury, how long will it take this Planet Mercury to revolve round the Sun?

Again let x (in years) represent how long it will take Mercury.

Then; $\left(\frac{x}{1}\right)^2 = \left(\frac{4}{10}\right)^3 \Rightarrow \quad x^2 = (0.4)^3 = 0.064 \Rightarrow x = 0.252\ years$

$$\Rightarrow x = (0.252 \times 365) days = 91\ days$$

It is pertinent to conclude that a model often helps to explain a system and to study the efforts of different components, and to make predictions about quantitative behaviours within the system.

E. Clearly, One *Real World Meaning of R*esult is exemplified in elliptical orbits of Planets instead of circular orbits that cannot explain why Planets orbited the Sun at different speed at different times. A circular orbit will only allow Planets to produce uniform motion, which is not a true reflection of physical observation.

That is, the Planet's motion (velocity) which is continuously changing through its elliptical orbit gives interpretation of these Kepler laws (especially law 2) to fit the real world meaning of results as in slow and fast movement of Planets around the orbit that validate real situation.

Furthermore, it is noted that, Kepler's laws and Newton's laws taken together implies the FORCE that holds the Planets in their elliptical orbits is

(1) directed towards the Sun from the Planet.
(2) proportional to the product of masses of the Sun (M) and the Planet (m).
(3) inversely proportional to the square of the distance (r) between the planet and the Sun.

This Force is the gravitational force, with G (Universal gravitational constant) as the constant of proportionality.

$$F \propto \frac{Mm}{r^2} \Rightarrow F = \frac{GMm}{r^2}$$

This Newton's law of Motion, with a gravitational force used in the 2^{nd} law, implies Kepler's laws. Hence, the Planets obey the same laws of motion as objects on the surface of the earth. These celestial laws of motion in space among the Planets derived by Copernicus and Kepler and that of terrestrial laws of motion among objects on Earth derived by Newton are congruent. Clearly, the motion of all solid objects and fluids either in heaven or earth can be described using the verifying language of Mathematics (Cha and Dym, 2000).

2.1 Applications of Results

Let us consider three cases.

a) Circumference of Planets

Let us consider Planet Earth. It is generally believed that students of Mathematical Sciences of today are familiar with the concept of longitudes and latitudes; hence should be able to calculate the circumference of the Earth.

For instance, if one locates Calabar and Kano, a distance 800km apart on the map of Nigeria; these two towns are on the same longitude. That is, Kano is directly due north of Calabar.

Also, the latitudes are 5^0 and 12^0 for Calabar and Kano respectively, a difference of 7^0.

That is, when the sun is overhead at Calabar, the direction of the Sun at Kano is 7^0 ie $<DAS' = 7^0$

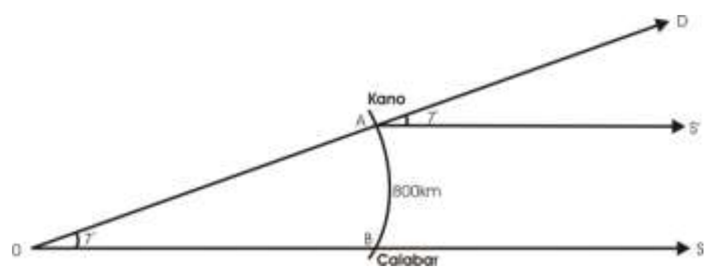
But $<AOB = <DAS'$ (Cor < s)

$\therefore <AOB = 7^0$

It follows then that 7^0 subtends 800km

Hence 360° will subtend $\frac{800\times360}{7}$km

i.e 41, 000km



$\therefore$The Circumference of the Earth is 41, 000km

It is very interesting according to Adetula (2005), to note that Newton proved mathematically that if this distance (41, 000km) can be covered in one hour, for an object projected to the sky, then the speed will represent the (minimum) escape velocity of the object to the outer space.

Actually, Newton proved that if an object is thrown with a speed

- Less than 24, 000km/hr into the air, it will always return to the Earth.
- Between 24, 000km/hr and 40, 000km/hr; it will form a satellite.
  However, more than 40, 000km/hr, it will escape from the influence of the Earth gravitation attraction.

These impeccable mathematical conclusions were only tested 60 years ago when appropriate tools (technology) were devised to move object with a speed greater than 24, 000km/hr into the air.

With this technology, the adage that says "what goes up must come down" was broken (Adetula, op.cit.).

b) Diameter of Planets

Let us be restricted to the diameter of the Planet Earth.

We know from Mathematics that the circumference of a circle (denoted by ◯ce) is directly propotional to the Diameter (D) of the circle.

$i.e.$ ◯ce α D

$i.e.$ ◯ce = kD (where k is the constant of proportionality, denoted by pi or $\pi = \frac{22}{7}$) $i.e.$ k = $\pi = \frac{22}{7}$

The ◯ce of the Planet Earth has just been calculated to be 41, 000km

∴ D, Diameter of Earth = $\frac{\text{Oce}}{\pi} = 41,000 \times \frac{7}{22} \simeq 13,000\text{km}$

This figure 13, 000km agrees perfectly well with the figure when diameter of the earth is google. Showing the model is flawless and indubitable (Adetula, 2005).

c) Model of pi(π) in the Bible

1Kings 7:23: *And he made a molten sea, ten cubits from the one brim to the other. It was round all about, and its height was five cubits. A line was of thirty cubits did encompass it round about.*

The "Molten Sea" is a water container of depth five cubits, probably for ablution.

The interesting part as a Mathematician is that its upper rim is described to be circular in shape with a diameter of ten cubits and circumference of thirty cubits.

Here π (pi), the ratio of the circumference to the diameter is given as $\frac{30}{10} = 3$ (instead of say $\frac{31\frac{1}{2}}{10} = 3.15$, which is very close to the actual value of π = 3.14…).

Clearly, the ratio of the circumference of a circle to its diameter is the same for all circles as earlier stated, and that it is slightly more than 3, was known to ancient Egyptian, Babylonian, Indian and Greek geometers as date back to around 2000 BC. However, over 1, 400 years later, the Book of Kings (600 BC), revealed that π = 3, which is not true (Adetula, 2012).

Clearly, using π = 3 to calculate the diameter of Planet Earth will lead to a wrong result in case of (b) above.

2.2 Mathematical Modeling as a Tool to Problem Solving

Mathematical Modeling is commonly regarded as the art of applying mathematics to real world problem with a view to better understand the problem. To some extent, Mathematical Modeling is not only related but a cognitive tool to problem solving.

2.2.1 Consider this Word Problem in Mathematics:

A farmer has hens and goats. These animals have 30 heads and 102 feet. How many hens and goats have the farmer?

One way to solve this problem, is to invoke Polya 4 – stage model thus;

- Understand the problem
- Devising a plan
- Carry out the plan
- Look back.

Understanding the problem: that involves hens, goats. Their heads and legs and the relationships among the involved items.

Devising a plan:

Let x stands for the number of hens.

Let y stands for the number of goats.

Take note of the meaning of the problem's information before translating the information from the problem into Mathematical Statement. Adetula (1988).

Understanding the Information	Problem Information	Mathematical Translation
If a hen has 1 head, the total number of hen's head is x, if a goat has 1 head, total number of goats' head is y	The animals have 30 heads	$x + y = 30$
Also, if a hen has 2 legs, the total number of hen's legs is $2x$ and if a goat has 4 legs, the total number of goat's legs is $4y$.	The animals have 102 feet	$2x + 4y = 102$

Carry out the plan -

The two equations which are mathematical representation of the problem are $x + y = 30$ and $2x + 4y = 102$.

These equations are solved simultaneously to obtain $x = 9$ and $y = 21$

That is, the number of hens is 9 and that of goats is 21.

Look back -

To check that these numbers are the correct solution, follow the written information using 9 for hen's number and 21 as goats' number to validate that the answers are correct. (Adetula, 2012)

2.2.2 Consider Another Example on Population Biology.

Population biology has to do with counting, estimating and predicting population size. The problems involved how to determine the mechanisms that cause and also maintain biological rhythms and also problems of management of exhaustible resources like timber, fish, or the spread of forest disease. (Turching, 2003).

It is noteworthy that population problems further led to developments in Mathematics models such as theories of logarithmic functions, probability, dynamical systems and wave propagation.

Now, consider this example below of predicting population size:

If the biologist group of Achievers University that have been studying the growth of bacteria population discovered that the rate of growth (change) of the population is directly proportional to the population. Assume that initial count bacteria is 1000 and after one hour, the count is 5000. Find the number of bacteria present immediately after 6 hours.

Let p represents the population.

Given that rate of growth of the population is directly proportional to the population,

$$\Longrightarrow \quad \frac{dp}{dt} \propto p$$

$$\Longrightarrow \quad \frac{dp}{dt} = kp \quad \text{(k, constant of proportionality)}$$

$$\therefore \quad \int \frac{dp}{p} = \int kdt \Longrightarrow \log_e p = kt + c$$

$$\therefore \quad p(t) = e^{kt+c} = e^{kt}.e^{c}$$

At initial count, i.e $t = 0$, then $p(0) = e^{c} = 1000$, Then $p(t) = e^{kt}.e^{c}$ becomes

$\therefore \quad p(t) = p(0)e^{kt} = 1000e^{kt}$

$p(1) = p(0)e^{kt} = 1000e^{k} = 5000$

$\therefore \quad e^{k} = 5 \Rightarrow k = \log_e 5$

$\therefore \quad p(t) = p(0)e^{kt} = 1000e^{(\log_e 5)t}$

Hence $p(6) = 1000e^{(\log_e 5)6} = 1000(5)^6 = 15.625 \times 10^6$

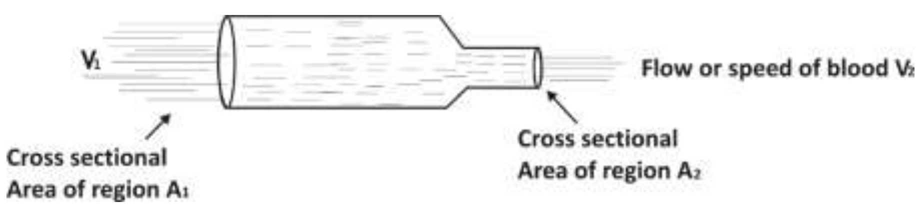
From the above exercise, it is clear that mathematical model is good for accurate prediction.

2.3 Illustration in Health Sciences

Computer solutions of partial Differential Equations (PDEs) arising in physiological fluid dynamics has various applications especially computational models of the heart, kidney, pancreas. It also allows the study of blood clotting, blood flow in the heart or arteries and veins, and air flow in the lungs or wave propagation of the inner ear can also be governed by coupled equations of motion. Simple Harmonic Motion (Differential Equations) exemplifies the rhythm of breathing of human being. (Curson and Cobelli, 2001).

In particular, blood flow is a study of measuring the blood pressure and finding the flow through the blood vessel. The model approach is to use the two-dimensional Navier-Strokes equation and continuity equations for fluid in cylindrical co-ordinate.

Actually, Navier-Strokes equation is a Partial DE which describe the motion of viscous fluid substance named after Physicist Navier and Mathematician Stroke and continuity equation or transport equation is an equation that describe the transport of some quantity. E.g. blood flow through a blood vessel as earlier mentioned. Since the blood vessel is not (necessarily) uniformly cylindrical, as depicted:



It follows therefore from fluid mechanics that $A_1V_1 = A_2V_2$ (conservation of mass). (Arnoild, 2004)

2.4 For Weather Prediction and Traffic Flow

Equation system in the hydrodynamic atmospheric models is used in weather prediction and forcasting. Officers in the metrological stations should be versed in these models. (Cross and Moczardin 1985).

For traffic flow, an equation system especially nonlinear hyperbolic Differential equations for function of two independent variables (that is, taking into account, two parameters) are used; these are exemplified in traffic flow speed, and traffic flow density. These parameters are calculated on each point of the road and information on the previous and next point of the road mesh is considered (Cross and Moscardin op.cit.).

3.0 Conclusion

From all intents and purposes, Mathematical modeling – a language that explains the real world, is all about *how to transform the information in the real world situation to Mathematical representation of the problem.* This is distinctively exemplified in the farmer/goat/hen problem earlier presented in this paper.

It is well known that a good mathematical model is not necessarily complete, because no model includes every aspects of real world. Therefore, models are more simplified or more idealised form of the world situation or system.

Today, some examples of good mathematical models with potential and positive impact are well documented to solve specific and concomitant problems, some of which are briefly explicated here: For weather, the objective is to predict and forecast the future; For flight simulation, the objective is to train especially in Aviation Schools; For Nuclear Arms race, the objective is to engage in strategy development especially World Superpowers; For traffic flow, the objective is to engage in regulation of traffic flow, etcetera. These examples are merely representative but not exhaustive.

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